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OPEN QUANTUM SYSTEM OF TWO COUPLED HARMONIC OSCILLATORS

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On the basis of the theory of Lindblad for open quantum systems we derive master equations for a system consisting of two harmonic oscillators. The time-dependence of expectation values, Wigner-function and Weyl operator are obtained and discussed. The chosen system can be applied for the description of the charge and mass asymmetry degrees of freedom in deep inelastic collisions in nuclear physics.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Открытая квантовая система для двух связанных гармонических осцилляторов

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В рамках теории Линдблада для открытых квантовых систем были получены уравнения мастера для двух связанных гармонических осцилляторов. Для этой системы, которая может описывать передачу массы и заряда в глубоконеупругих столкновениях, были получены в явном виде средние величины, функция Вагнера и оператор Вайла.

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1. *Introduction*

In a recent paper /1/ it was shown that various master equations for the damped quantum oscillator used in the literature for the description of damped collective modes in deep inelastic collisions in nuclear physics are particular cases of the master equation derived by Lindblad /2,3/.

In the present paper we extend our previous work /4/ on the dynamics of charge and mass equilibration in deep

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inelastic collisions by describing the corresponding collective modes as two coupled damped quantum oscillators. The damping of these oscillators can be carried out by the method of Lindblad [2,3]. In Sect.2 we present the equation of motion of the open quantum system in the Heisenberg picture. With this equation we derive the time-dependence of the expectation values of the coordinates and momenta and their variances, as shown in Sect.3. The connections with the Wigner-function and Weyl operator are discussed in Sect.4. Finally, in Sect.5, we demonstrate the time-dependence of the various quantities for a simplified version of the model, where the decay constants can be calculated analytically.

2. *The Equation of Motion in the Heisenberg Picture*

If $\tilde{\phi}_t$ is the dynamical semigroup describing the time evolution of the open quantum system in the Heisenberg picture, then the master equation is given for an operator A as follows [2,3]:

$$\frac{d\tilde{\phi}_t(A)}{dt} = \tilde{L}(\tilde{\Phi}_t(A)) = \frac{i}{\hbar}[H,\tilde{\phi}_t(A)] + \frac{1}{2\hbar}\sum_j (V_j^*[\tilde{\Phi}_t(A),V_j]+[V_j^*,\tilde{\phi}_t(A)]V_j). \qquad (1)$$

The operators H, V_j, V_j^* $(j = 1, 2, 3, 4)$ are taken to be functions of the basic observables of the two quantum oscillators. The coordinates are q_1 and q_2, and the momenta p_1 and p_2 obeying the usual commutation relations

$$[q_1, p_1] = i\hbar I, \quad [q_2, p_2] = i\hbar I,$$

$$[q_1, q_2] = 0, \quad [p_1, p_2] = 0, \; [q_1, p_2] = 0, \; [q_2, p_1] = 0.$$

In order to obtain an analytically solvable model, H is taken to be a polynomial of second degree in these basic observables and V_j, V_j^* are taken to be polynomials of first degree. Then in the linear space spanned by q_1, q_2, p_1, p_2, there exist only four linearly independent operators $V_{j=1,2,3,4}$:

$$V_j = \sum_{\kappa=1}^{2} a_{j\kappa} p_\kappa + \sum_{\kappa=1}^{2} b_{j\kappa} q_\kappa , \qquad (2)$$

where $a_{j\kappa}, b_{j\kappa} \in \mathbb{C}$ with $j = 1, 2, 3, 4$, and $\kappa = 1, 2$. Then it yields

$$V_j^* = \sum_{\kappa=1}^{2} a_{j\kappa}^* p_\kappa + \sum_{\kappa=1}^{2} b_{j\kappa}^* q_\kappa , \qquad (3)$$

where $a^*_{j\kappa}$, $b^*_{j\kappa}$ are the complex conjugates of $a_{j\kappa}$, $b_{j\kappa}$. The Hamiltonian H is chosen in the form of two coupled oscillators

$$H = \sum_{\kappa=1}^{2} \left(\frac{1}{2m_\kappa} p_\kappa^2 + \frac{m_\kappa \omega_\kappa^2}{2} q_\kappa^2\right) + \kappa_{12} p_1 p_2 +$$
$$+ \frac{1}{2} \sum_{\kappa_1,\kappa_2=1}^{2} \mu_{\kappa_1\kappa_2}(p_{\kappa_1} q_{\kappa_2} + q_{\kappa_2} p_{\kappa_1}) + \nu_{12} q_1 q_2 . \tag{4}$$

Inserting the Hamiltonian H and the operators V_j and V_j^* into Eq.(1) we obtain

$$\tilde{L}(A) = \tilde{L}_1(A) + \tilde{L}_2(A) + \tilde{L}_{12}(A) , \tag{5}$$

where $\tilde{L}_1, \tilde{L}_2$ and $\tilde{L}_{12}$ are given as (κ = 1, 2):

$$\tilde{L}_\kappa(A) = \frac{i}{\hbar}[H_{o\kappa}, A] - \frac{1}{\hbar^2} D_{p_\kappa p_\kappa}[q_\kappa,[q_\kappa, A]] - \frac{1}{\hbar^2} D_{q_\kappa q_\kappa}[p_\kappa,[p_\kappa, A]] +$$
$$+ \frac{1}{\hbar^2} D_{p_\kappa q_\kappa}[q_\kappa,[p_\kappa, A]] + \frac{1}{\hbar^2} D_{q_\kappa p_\kappa}[p_\kappa,[q_\kappa, A]] +$$
$$+ \frac{i}{2\hbar}(\lambda_{\kappa\kappa} - \mu_{\kappa\kappa})([A, p_\kappa]q_\kappa + q_\kappa[A, p_\kappa]) - \tag{6}$$
$$- \frac{i}{2\hbar}(\lambda_{\kappa\kappa} + \mu_{\kappa\kappa})([A, q_\kappa]p_\kappa + p_\kappa[A, q_\kappa]) ,$$

$$\tilde{L}_{12}(A) = - \frac{1}{\hbar^2} D_{p_1 p_2}([q_1,[q_2, A]] + [q_2,[q_1, A]]) -$$
$$- \frac{1}{\hbar^2} D_{q_1 q_2}([p_1,[p_2, A]] + [p_2,[p_1, A]]) +$$
$$+ \frac{1}{\hbar^2} D_{p_1 q_2}([q_1,[p_2, A]] + [p_2,[q_1, A]]) +$$
$$+ \frac{1}{\hbar^2} D_{q_1 p_2}([p_1,[q_2, A]] + [q_2,[p_1, A]]) +$$
$$+ \frac{i}{2\hbar}(\alpha_{12} - \kappa_{12})([A, p_1]p_2 + p_2[A, p_1]) -$$
$$- \frac{i}{2\hbar}(\alpha_{12} + \kappa_{12})([A, p_2]p_1 + p_1[A, p_2]) +$$

$$+ \frac{i}{2\hbar}(\beta_{12} - \nu_{12})([A, q_1]q_2 + q_2[A, q_1]) -$$

$$- \frac{i}{2\hbar}(\beta_{12} + \nu_{12})([A, q_2]q_1 + q_1[A, q_2]) +$$

$$+ \frac{i}{2\hbar}(\lambda_{12} - \mu_{12})([A, p_1]q_2 + q_2[A, p_1]) -$$

$$- \frac{i}{2\hbar}(\lambda_{12} + \mu_{12})([A, q_2]p_1 + p_1[A, q_2]) +$$

$$+ \frac{i}{2\hbar}(\lambda_{21} - \mu_{21})([A, p_2]q_1 + q_1[A, p_2]) -$$

$$- \frac{i}{2\hbar}(\lambda_{21} + \mu_{21})([A, q_1]p_2 + p_2[A, q_1]). \qquad (7)$$

Here, we used the following abbreviations ($\kappa = 1, 2$):

$$H_{o\kappa} = \frac{1}{2m_\kappa}p_\kappa^2 + \frac{m_\kappa \omega_\kappa^2}{2}q_\kappa^2, \quad D_{q_\kappa q_\mu} = D_{q_\mu q_\kappa} = \frac{\hbar}{2}\mathrm{Re}(\vec{a}_\kappa^* \vec{a}_\mu),$$

$$D_{p_\kappa p_\mu} = D_{p_\mu p_\kappa} = \frac{\hbar}{2}\mathrm{Re}(\vec{b}_\kappa^* \vec{b}_\mu), \quad D_{q_\kappa p_\mu} = D_{p_\mu q_\kappa} = -\frac{\hbar}{2}\mathrm{Re}(\vec{a}_\kappa^* \vec{b}_\mu), \quad (8)$$

$$\alpha_{12} = -\alpha_{21} = -\mathrm{Im}(\vec{a}_1^* \vec{a}_2), \quad \beta_{12} = -\beta_{21} = -\mathrm{Im}(\vec{b}_1^* \vec{b}_2), \quad \lambda_{\kappa\mu} = -\mathrm{Im}(\vec{a}_\kappa^* \vec{b}_\mu).$$

The scalar products are formed with the vectors $\vec{a}_\kappa$, $\vec{b}_\kappa$ and their complex conjugates $\vec{a}_\kappa^*, \vec{b}_\kappa^*$. The vectors have the components

$$\vec{a}_\kappa = (a_{1\kappa}, a_{2\kappa}, a_{3\kappa}, a_{4\kappa}), \quad \vec{b}_\kappa = (b_{1\kappa}, b_{2\kappa}, b_{3\kappa}, b_{4\kappa}). \qquad (9)$$

Now, as a consequence of the definitions (8) of the phenomenological constants which appear in L(A) and of the positivity of the matrix formed by the four vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{b}_1$, $\vec{b}_2$, it follows that the principal minors of this matrix are positive or zero. This matrix is given by

$$\frac{\hbar}{2}\begin{pmatrix} \vec{a}_1^*\vec{a}_1 & \vec{a}_1^*\vec{a}_2 & \vec{a}_1^*\vec{b}_1 & \vec{a}_1^*\vec{b}_2 \\ \vec{a}_2^*\vec{a}_1 & \vec{a}_2^*\vec{a}_2 & \vec{a}_2^*\vec{b}_1 & \vec{a}_2^*\vec{b}_2 \\ \vec{b}_1^*\vec{a}_1 & \vec{b}_1^*\vec{a}_2 & \vec{b}_1^*\vec{b}_1 & \vec{b}_1^*\vec{b}_2 \\ \vec{b}_2^*\vec{a}_1 & \vec{b}_2^*\vec{a}_2 & \vec{b}_2^*\vec{b}_1 & \vec{b}_2^*\vec{b}_2 \end{pmatrix} =$$

$$
=\begin{pmatrix}
D_{q_1q_1} & D_{q_1q_2}-\frac{i\hbar}{2}\alpha_{12} & -D_{q_1p_1}-\frac{i\hbar}{2}\lambda_{11} & -D_{q_1p_2}-\frac{i\hbar}{2}\lambda_{12} \\
D_{q_2q_1}-\frac{i\hbar}{2}\alpha_{21} & D_{q_2q_2} & -D_{q_2p_1}-\frac{i\hbar}{2}\lambda_{21} & -D_{q_2p_2}-\frac{i\hbar}{2}\lambda_{22} \\
-D_{p_1q_1}+\frac{i\hbar}{2}\lambda_{11} & -D_{p_1q_2}+\frac{i\hbar}{2}\lambda_{21} & D_{p_1p_1} & D_{p_1p_2}-\frac{i\hbar}{2}\beta_{12} \\
-D_{p_2q_1}+\frac{i\hbar}{2}\lambda_{12} & -D_{p_2q_2}+\frac{i\hbar}{2}\lambda_{22} & D_{p_2p_1}-\frac{i\hbar}{2}\beta_{21} & D_{p_2p_2}
\end{pmatrix} \tag{10}
$$

For example, we derive the following condition from the positivity of (10):

$$
D_{q_1q_1}D_{q_2q_2}-(D_{q_1q_2})^2 \geq \frac{\hbar^4}{4}\alpha_{12}^2 . \tag{11}
$$

This inequality and the corresponding ones derived from Eq.(10) are constraints imposed on the phenomenological constants by the fact that $\tilde{\phi}_t$ is a dynamical semigroup[2,3].

3. *The Time-Dependence of Expectation Values*

The time-dependent expectation values of selfadjoint operators A and B can be written with the density operator ρ, describing the initial state of the quantum system, as follows:

$$
m_A(t) = \mathrm{Tr}(\rho\tilde{\phi}_t(A)), \quad \sigma_{AB}(t) = \frac{1}{2}\mathrm{Tr}(\rho\tilde{\phi}_t(AB+BA)) . \tag{12}
$$

In the following we denote the vector with the four components $m_{q_1}(t)$, $m_{q_2}(t)$, $m_{p_1}(t)$ and $m_{p_2}(t)$ by $\vec{m}(t)$ and the following 4x4 matrix by $\hat{\sigma}(t)$:

$$
\hat{\sigma}(t) = \begin{pmatrix}
\sigma_{q_1q_1} & \sigma_{q_1q_2} & \sigma_{q_1p_1} & \sigma_{q_1p_2} \\
\sigma_{q_2q_1} & \sigma_{q_2q_2} & \sigma_{q_2p_1} & \sigma_{q_2p_2} \\
\sigma_{p_1q_1} & \sigma_{p_1q_2} & \sigma_{p_1p_1} & \sigma_{p_1p_2} \\
\sigma_{p_2q_1} & \sigma_{p_2q_2} & \sigma_{p_2p_1} & \sigma_{p_2p_2}
\end{pmatrix} \tag{13}
$$

Then via direct calculation of $\tilde{L}(q_\kappa)$ and $\tilde{L}(p_\kappa)$ we obtain

$$
\frac{d\vec{m}}{dt} = \hat{Y}\vec{m} , \tag{14}
$$

where

$$\hat{Y} = \begin{pmatrix} -\lambda_{11} + \mu_{11} & -\lambda_{12} + \mu_{12} & 1/m_1 & -\alpha_{12} + \kappa_{12} \\ -\lambda_{21} + \mu_{21} & -\lambda_{22} + \mu_{22} & \alpha_{12} + \kappa_{12} & 1/m_2 \\ -m_1 \omega_1^2 & \beta_{12} - \nu_{12} & -\lambda_{11} - \mu_{11} & -\lambda_{21} - \mu_{21} \\ -\beta_{12} - \nu_{12} & -m_2 \omega_2^2 & -\lambda_{12} - \mu_{12} & -\lambda_{22} - \mu_{22} \end{pmatrix} \tag{15}$$

From Eq.(14) it follows that

$$\vec{m}(t) = \hat{M}(t)\,\vec{m}(0) = \exp(t\hat{Y})\vec{m}(0) \,, \tag{16}$$

where $\vec{m}(0)$ is given by the initial conditions. The matrix $\hat{M}(t)$ has to fulfil the condition

$$\lim_{t\to\infty} \hat{M}(t) = 0 \,. \tag{17}$$

In order that this limit exists, $\hat{Y}$ must have only eigenvalues with negative real parts.

By direct calculation of $\tilde{L}(q_\kappa q_\mu)$, $\tilde{L}(p_\kappa p_\mu)$ and $\tilde{L}(q_\kappa p_\mu + p_\mu q_\kappa)$, $\kappa, \mu = 1,2$, we obtain

$$\frac{d\hat{\sigma}}{dt} = \hat{Y}\hat{\sigma} + \hat{\sigma}\hat{Y}^T + 2\hat{D} \,, \tag{18}$$

where $\hat{D}$ is the matrix of the diffusion coefficients

$$\hat{D} = \begin{pmatrix} D_{q_1q_1} & D_{q_1q_2} & D_{q_1p_1} & D_{q_1p_2} \\ D_{q_2q_1} & D_{q_2q_2} & D_{q_2p_1} & D_{q_2p_2} \\ D_{p_1q_1} & D_{p_1q_2} & D_{p_1p_1} & D_{p_1p_2} \\ D_{p_2q_1} & D_{p_2q_2} & D_{p_2p_1} & D_{p_2p_2} \end{pmatrix} ; \tag{19}$$

and $\hat{Y}^T$, the transposed matrix of $\hat{Y}$. The time-dependent solution of Eq.(18) can be written as

$$\hat{\sigma}(t) = \hat{M}(t)\,(\hat{\sigma}(0) - \hat{\Sigma})\,\hat{M}^T(t) + \hat{\Sigma} \,, \tag{20}$$

where $\hat{M}(t)$ is defined in Eq.(16). The matrix $\hat{\Sigma}$ is time-independent and solves the static problem of Eq.(18) $(d\hat{\sigma}/dt = 0)$:

$$\hat{Y}\hat{\Sigma} + \hat{\Sigma}\hat{Y}^T + 2\hat{D} = 0 \,. \tag{21}$$

Now we assume that the following limes exist for $t \to \infty$:

$$\hat{\sigma}(\infty) = \lim_{t\to\infty} \hat{\sigma}(t) \,. \tag{22}$$

In that case it follows from (20) with Eq.(17):

$$\hat{\sigma}(\infty) = \hat{\Sigma} \,. \tag{23}$$

Inserting Eq.(23) into Eq.(20) we obtain the basic equations for our purposes:

$$\hat{\sigma}(t) = \hat{M}(t)(\hat{\sigma}(0) - \hat{\sigma}(\infty))\hat{M}^T(t) + \hat{\sigma}(\infty), \quad (24)$$

where

$$\hat{Y}\hat{\sigma}(\infty) + \hat{\sigma}(\infty)\hat{Y}^T = -2\hat{D}. \quad (25)$$

4. The Wigner-Function and Weyl Operator

Finally we want to discuss the time-dependence of the Wigner-function. This function is defined as

$$f(x_1, x_2, y_1, y_2, t) = \frac{1}{(2\pi\hbar)^4} \iiiint_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}(x_1\eta_1 + x_2\eta_2 - y_1\xi_1 - y_2\xi_2)\right)\times$$
$$\times \mathrm{Tr}(\rho\tilde{\phi}_t(W(\xi_1,\xi_2;\eta_1,\eta_2))d\xi_1 d\xi_2 d\eta_1 d\eta_2, \quad (26)$$

where the Weyl operator W is defined by ($\xi_1, \xi_2, \eta_1, \eta_2$ real):

$$W(\xi_1,\xi_2;\eta_1,\eta_2) = \exp\left(\frac{i}{\hbar}(\eta_1 q_1 + \eta_2 q_2 - \xi_1 p_1 - \xi_2 p_2)\right). \quad (27)$$

Using the method developed by Lindblad [2,3] for the one-dimensional case we find for the time-development of the Weyl operator the relation

$$\tilde{\phi}_t(W(\xi_1,\xi_2;\eta_1,\eta_2)) = W(\xi_1(t),\xi_2(t);\eta_1(t),\eta_2(t))\exp(g(t)). \quad (28)$$

The real functions $\vec{\xi}(t) = (\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t))$ and $g(t)$ satisfy the equation of motion:

$$\frac{d\vec{\xi}(t)}{dt} = \hat{J}\hat{Y}^T\hat{J}^{-1}\vec{\xi}(t), \quad (29)$$

$$\frac{dg(t)}{dt} = \frac{1}{\hbar^2}\vec{\xi}(t)\hat{J}\hat{D}\hat{J}^{-1}\vec{\xi}(t), \quad (30)$$

where

$$\hat{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (31)$$

Eqs. (29) and (30) are obtained by inserting the Weyl operator $W(\xi_1,\xi_2;\eta_1,\eta_2)$ into the equation of motion (Eq.(1)) with L defined in Eqs.(5), (6) and (7). The initial conditions for the coordinates $\xi_1(t)$, $\xi_2(t)$, $\eta_1(t)$ and $\eta_2(t)$ are determined by $\xi_2(0) = \xi_1$, $\xi_2(0) = \xi_2$, $\eta_1(0) = \eta_1$ and $\eta_2(0) = \eta_2$, respectively, and $g(t)$ by $g(0) = 0$. From Eqs.(29) and (30) we find that $\vec{\xi}(t)$ is a linear function in the coordinates ξ_1, ξ_2, η_1 and η_2 and $g(t)$ a quadratic function.

The Weyl operator can be used to calculate the time-dependent expectation values $\vec{m}(t)$ and $\hat{\sigma}(t)$ (see Eqs.(16) and (20)), since this operator is connected with the coordinates and momenta via the derivatives:

$$\frac{\partial W}{\partial \xi_i}\Big|_{\vec{\xi}=0} = -\frac{i}{\hbar}p_i\,, \quad \frac{\partial W}{\partial \eta_i}\Big|_{\vec{\xi}=0} = \frac{i}{\hbar}q_i\,, \quad \frac{\partial^2 W}{\partial \xi_i \partial \xi_j}\Big|_{\vec{\xi}=0} = -\frac{1}{\hbar^2}p_i p_j\,,$$

$$\frac{\partial^2 W}{\partial \xi_i \partial \eta_j}\Big|_{\vec{\xi}=0} = \frac{1}{2\hbar^2}(p_i q_j + q_j p_i)\,, \quad \frac{\partial^2 W}{\partial \eta_i \partial \eta_j}\Big|_{\vec{\xi}=0} = -\frac{1}{\hbar^2}q_i q_j. \tag{32}$$

For example, one obtains by using Eq.(32)

$$\sigma_{p_i p_j}(t) = -\hbar^2 \mathrm{Tr}\Big(\rho \frac{\partial^2 \tilde{\phi}_t(W)}{\partial \xi_i(0)\,\partial \xi_j(0)}\Big|_{\vec{\xi}(t=0)=0}\Big). \tag{33}$$

Equations of this type can be evaluated with the help of Eqs.(28)-(30) and lead to the same results for $\vec{m}(t)$ and $\hat{\sigma}(t)$ as given in Sect.3. With the Weyl-opeartor (28) we can calculate the time-development of the Wigner-function. For this purpose we use the Fourier transformed of the Wigner-function at t = 0:

$$\begin{aligned}\mathrm{Tr}\Big(\rho \exp\big(\tfrac{i}{\hbar}(\eta_1' q_1 + \eta_2' q_2 - \xi_1' p_1 - \xi_2' p_2)\big)\Big) &= \\ = \iiiint_{-\infty}^{+\infty} \exp\big(\tfrac{i}{\hbar}(x_1\eta_1' + x_2\eta_2' - y_1\xi_1' - y_2\xi_2')\big) &\times \\ \times f(x_1, x_2, y_1, y_2, t=0)\,dx_1\,dx_2\,dy_1\,dy_2 &.\end{aligned} \tag{34}$$

When this relation is inserted into Eq.(26) after the Weyl-operator $\tilde{\phi}_t(W)$ is expressed by (28), one can integrate over the coordinates ξ_1, ξ_2, η_1 and η_2 with the following result for the Wigner-function:

$$f(x_1,x_2,y_1,y_2,t) = \frac{1}{\sqrt{\det(4\pi\hat{Z})}} \iiiint_{-\infty}^{+\infty} \exp\big(-\tfrac{1}{4}(\vec{x} - \vec{x}'\hat{M}^T)\hat{Z}^{-1}(x - \hat{M}x')\big) \times$$

$$\times f(x_1', x_2', y_1', y_2', t=0)\,dx_1'\,dx_2'\,dy_1'\,dy_2'\,, \tag{35}$$

where $\vec{x} = (x_1, x_2, y_1, y_2)$ and the macrix $\hat{Z}(t)$ is given by

$$\hat{Z}(t) = \int_0^t \hat{M}(t')\,\hat{D}\hat{M}^T(t')\,dt'. \tag{36}$$

This definition can be applied in order to rewrite Eq.(24):

$$\hat{\sigma}(t) = \hat{M}(t)\,\hat{\sigma}(0)\,\hat{M}^T(t) + 2\hat{Z}(t)\,. \tag{37}$$

In the particular case when we set

$$f(x_1,x_2,y_1,y_2,t=0) = \frac{1}{\sqrt{\det(2\pi\hat{\sigma}(0))}} \exp\big(-\tfrac{1}{2}(\vec{x} - \vec{m}(0))\hat{\sigma}(0)^{-1}(\vec{x} - \vec{m}(0))\big)\,, \tag{38}$$

we obtain from Eq.(35)

$$f(x_1, x_2, y_1, y_2, t) = \frac{1}{\sqrt{\det(2\pi\hat{\sigma}(t))}} \exp(-\frac{1}{2}(\vec{x}-\vec{m}(t))\,\hat{\sigma}(t)^{-1}(\vec{x}-\vec{m}(t))), \tag{39}$$

which is the well-known result for Wigner-functions[5,6,7].

5. *Example for Damped Oscillators*

In order to illustrate the formalism developed in the preceding Sections, we present a simple example of two oscillators, which are not directly coupled, i.e. $\kappa_{12}=0$ $\mu_{ij}=0$, $\nu_{12}=0$. In this case the matrix $\hat{Y}$, governing the time-development of the expectation values $\vec{m}(t)$ and $\hat{\sigma}(t)$, becomes

$$\hat{Y} = \begin{pmatrix} -\lambda_{11} & -\lambda_{12} & 1/m_1 & -\alpha_{12} \\ -\lambda_{21} & -\lambda_{22} & \alpha_{12} & 1/m_2 \\ -m_1\omega_1^2 & \beta_{12} & -\lambda_{11} & -\lambda_{21} \\ -\beta_{12} & -m_2\omega_2^2 & -\lambda_{12} & -\lambda_{22} \end{pmatrix}. \tag{40}$$

For the calculation of the matrix $\hat{M}(t)$ we must diagonalize the matrix $\hat{Y}$ by solving the corresponding secular equation, i.e. $\det(\hat{Y}-z\hat{I})=0$, where z is the eigenvalue and $\hat{I}$ the unit matrix. According to Eq.(40) one obtains an equation of 4th order for the eigenvalues z, which can be solved analytically only for special examples. Such an example is the particular case with $\alpha_{12}=0$, $\beta_{12}=0$ $\lambda_{12}=0$ and $\lambda_{21}=0$, where the secular equation is obtained as

$$((z+\lambda_{11})^2+\omega_1^2)\cdot((z+\lambda_{22})^2+\omega_2^2)=0. \tag{41}$$

The eigenvalues are

$$z_1=-\lambda_{11}+i\omega_1,\; z_2=-\lambda_{22}+i\omega_2,\; z_3=-\lambda_{11}-i\omega_1,\; z_4=-\lambda_{22}-i\omega_2. \tag{42}$$

Only positive values of λ_{11} and λ_{22} fulfil Eq.(17). Applying the eigenvalues z_i of $\hat{Y}$ we can write the time-dependent matrix $\hat{M}(t)$ as follows:

$$M_{mn}(t) = \sum_i N_{mi}\exp(z_i t)\,N^{-1}_{in}, \tag{43}$$

where the matrix $\hat{N}$ represents the eigenvectors of $\hat{Y}$,

$$\sum_n Y_{mn}N_{ni} = z_i N_{mi}. \tag{44}$$

In the case of the particular example with eigenvalues (42) the matrix $\hat{M}(t)$ is given by

$$\hat{M}(t) = \begin{pmatrix} \exp(-\lambda_{11}t)\cos\omega_1 t & 0 & \frac{1}{m_1\omega_1}\exp(-\lambda_{11}t)\sin\omega_1 t & 0 \\ 0 & \exp(-\lambda_{22}t)\cos\omega_2 t & 0 & \frac{1}{m_2\omega_2}\exp(-\lambda_{22}t)\sin\omega_2 t \\ -m_1\omega_1\exp(-\lambda_{11}t)\sin\omega_1 t & 0 & \exp(-\lambda_{11}t)\cos\omega_1 t & 0 \\ 0 & -m_2\omega_2\exp(-\lambda_{22}t)\sin\omega_2 t_2 & 0 & \exp(-\lambda_{22}t)\cos\omega_2 t \end{pmatrix} \quad (45)$$

We note that $\hat{M}(t)$ decays exponentially in time, if λ_{11} and λ_{22} are positive. The matrix $\hat{M}(t)$ can be used to evaluate $\hat{\sigma}(t)$ defined by Eqs.(24) or (37). For example we find the following expression for $\sigma_{12} = \sigma_{q_1q_2}$ with $\hat{M}(t)$ of Eq.(45):

$$\begin{aligned} \sigma_{q_1q_2}(t) &= \exp(-(\lambda_{11}+\lambda_{22})t) \times \\ &\times \{(\sigma_{q_1q_2}(0) - \sigma_{q_1q_2}(\infty))\cos\omega_1 t\cos\omega_2 t + \\ &+ \frac{1}{m_1\omega_1}(\sigma_{q_2p_1}(0) - \sigma_{q_2p_1}(\infty))\sin\omega_1 t\cos\omega_2 t + \\ &+ \frac{1}{m_2\omega_2}(\sigma_{q_1p_2}(0) - \sigma_{q_1p_2}(\infty))\cos\omega_1 t\sin\omega_2 t + \\ &+ \frac{1}{m_1m_2\omega_1\omega_2}(\sigma_{p_1p_2}(0) - \sigma_{p_1p_2}(\infty))\sin\omega_1 t\sin\omega_2 t\} + \sigma_{q_1q_2}(\infty). \end{aligned} \quad (46)$$

Similar expressions are found for the other matrix elements of $\hat{\sigma}(t)$. The matrix elements of $\hat{\sigma}(\infty)$ depend on $\hat{Y}$ and $\hat{D}$ and must be evaluated with Eq.(25) or by the relation

$$\hat{\sigma}(\infty) = 2\int_0^\infty \hat{M}(t')\hat{D}\hat{M}^T(t')\,dt'. \quad (47)$$

As an example we present the value of $\sigma_{q_1q_2}(\infty)$:

$$\begin{aligned} \sigma_{q_1q_2}(\infty) &= 2[((\lambda_{11}+\lambda_{22})^2 + (\omega_1+\omega_2)^2)((\lambda_{11}+\lambda_{22})^2 + (\omega_1-\omega_2)^2)]^{-1} \times \\ &\times \{(\lambda_{11}+\lambda_{22})((\lambda_{11}+\lambda_{22})^2 + \omega_1^2 + \omega_2^2)D_{q_1q_2} + ((\lambda_{11}+\lambda_{22})^2 + \omega_1^2 - \omega_2^2)D_{q_2p_1}/m_1 + \\ &+ ((\lambda_{11}+\lambda_{22})^2 + \omega_2^2 - \omega_1^2)D_{q_1p_2}/m_2 + 2(\lambda_{11}+\lambda_{22})D_{p_1p_2}/(m_1m_2)\}. \end{aligned} \quad (48)$$

Similar expressions are obtained for the other matrix elements of $\hat{\sigma}(\infty)$.

6. *Conclusions*

In this paper we have formulated the time-dependence of a system consisting of two damped oscillators. The equations of motion of operators and expectation values are obtained by applying the theory of Lindblad [2,3] and by extending it explicitly to two oscillators. The resulting time-dependence of the expectation values shows an exponential damping.

In order to apply this theory to the dynamics of the charge and mass asymmetry degrees of freedom in deep inelastic collisions one can start from the Hamiltonian H given by Gupta et al. [4], which depends on the coordinates $\eta_Z = (Z_1 - Z_2)/(Z_1+Z_2)$ and $\eta_N = (N_1-N_2)/(N_1+N_2)$ of the proton and neutron asymmerties, respectively. By comparing this Hamiltonian with Eq.(4) we can set $\kappa_{12}=0$ and $\mu_{ij}=0$ in Eq.(4). In a first application of this theory the matrix (10) which describes the damping and diffusion of the system can be chosen freely by fitting the experimental data. Work in this direction is in progress.

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